A De Bruijn–Erdös theorem for chordal graphs

Laurent Beaudou, Adrian Bondy, Xiaomin Chen, Ehsan Chiniforooshan, Maria Chudnovsky, Vasek Chvâtal, Nicolas Fraiman, Yori Zwols

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Laurent Beaudou (Université Blaise Pascal, Clermont-Ferrand)[1]
Adrian Bondy (Université Paris 6)[2]
Xiaomin Chen (Shanghai Jianshi LTD)[3]
Ehsan Chiniforooshan (Google, Waterloo)[4]
Maria Chudnovsky (Columbia University, New York)[5]
Vašek Chvátal (Concordia University, Montreal)[6]
Nicolas Fraiman (McGill University, Montreal)[7]
Yori Zwols (Concordia University, Montreal)[8]

Abstract

A special case of a combinatorial theorem of De Bruijn and Erdős asserts that every noncollinear set of n points in the plane determines at least n distinct lines. Chen and Chvátal suggested a possible generalization of this assertion in metric spaces with appropriately defined lines. We prove this generalization in all metric spaces induced by connected chordal graphs.

1 Introduction

It is well known that

(i) *every noncollinear set of n points in the plane determines at least n distinct lines.*

As noted by Erdős [11], theorem (i) is a corollary of the Sylvester–Gallai theorem (asserting that, for every noncollinear set S of finitely many points in

[1] `laurent.beaudou@ens-lyon.org`
[2] `adrian.bondy@sfr.fr`
[3] `gougle@gmail.com`
[4] `chiniforooshan@alumni.uwaterloo.ca`
[5] `mchudnov@columbia.edu`
Partially supported by NSF grants DMS-1001091 and IIS-1117631
[6] `chvatal@cse.concordia.ca`
Canada Research Chair in Combinatorial Optimization
[7] `nfraiman@gmail.com`
[8] `yzwols@gmail.com`

the plane, some line goes through precisely two points of S); it is also a special case of a combinatorial theorem proved later by De Bruijn and Erdős [10].

Theorem (i) involves neither measurement of distances nor measurement of angles: the only notion employed here is incidence of points and lines. Such theorems are a part of *ordered geometry* [7], which is built around the ternary relation of *betweenness*: point b is said to lie between points a and c if b is an interior point of the line segment with endpoints a and c. It is customary to write $[abc]$ for the statement that b lies between a and c. In this notation, a *line* $\overline{uv}$ is defined — for any two distinct points u and v — as

$$\{u, v\} \cup \{p : [puv] \vee [upv] \vee [uvp]\}. \tag{1}$$

In terms of the Euclidean metric *dist*, we have

$$\begin{aligned}&[abc] \Leftrightarrow \\ &a, b, c \text{ are three distinct points and } dist(a, b) + dist(b, c) = dist(a, c).\end{aligned} \tag{2}$$

In an arbitrary metric space, equivalence (2) defines the ternary relation of *metric betweenness* introduced in [12] and further studied in [1, 3, 8]; in turn, (1) defines the line $\overline{uv}$ for any two distinct points u and v in the metric space. The resulting family of lines may have strange properties. For instance, a line can be a proper subset of another: in the metric space with points u, v, x, y, z and

$$\begin{aligned}dist(u, v) = dist(v, x) = dist(x, y) = dist(y, z) = dist(z, u) = 1,\\ dist(u, x) = dist(v, y) = dist(x, z) = dist(y, u) = dist(z, v) = 2,\end{aligned}$$

we have

$$\overline{vy} = \{v, x, y\} \quad \text{and} \quad \overline{xy} = \{v, x, y, z\}.$$

Chen [4] proved, using a definition of $\overline{uv}$ different from (1), that the Sylvester–Gallai theorem generalizes in the framework of metric spaces. Chen and Chvátal [5] suggested that theorem (i), too, might generalize in this framework:

(ii) *True or false? Every metric space on n points, where $n \geq 2$, either has at least n distinct lines or else has a line that consists of all n points.*

They proved that

- every metric space on n points either has at least $\lg n$ distinct lines or else has a line that consists of all n points

and noted that the lower bound $\lg n$ can be improved to $\lg n + \frac{1}{2}\lg\lg n + \frac{1}{2}\lg\frac{\pi}{2} - o(1)$.

Every connected undirected graph induces a metric space on its vertex set, where $dist(u,v)$ is defined as the smallest number of edges in a path from vertex u to vertex v. Chiniforooshan and Chvátal [6] proved that

- every metric space induced by a connected graph on n vertices either has $\Omega(n^{2/7})$ distinct lines or else has a line that consists of all n vertices;

we will prove that the answer to (ii) is 'true' for all metric spaces induced by connected chordal graphs.

Theorem 1. *Every metric space induced by a connected chordal graph on n vertices, where $n \geq 2$, either has at least n distinct lines or else has a line that consists of all n vertices.*

For graph-theoretic terminology, we refer the reader to Bondy and Murty[2].

2 The proof

Given an undirected graph, let us write $[abc]$ to mean that a, b, c are three distinct vertices such that $dist(a,b) + dist(b,c) = dist(a,c)$; this is equivalent to saying that b is an interior vertex of a shortest path from a to c.

Lemma 1. *Let s, x, y be vertices in a finite chordal graph such that $[sxy]$. If $\overline{sx} = \overline{sy}$, then x is a cut vertex separating s and y.*

Proof. The set of all vertices u such that $dist(s,u) = dist(s,x)$ separates s and y. Among all its subsets that separate s and y, choose a minimal one and call it C. Since x is an interior vertex of a shortest path from s to y, it belongs to C. To prove that C includes no other vertex, assume, to the contrary, that C includes a vertex u other than x.

Our graph with C removed has distinct connected components S and Y such that $s \in S$ and $y \in Y$; the minimality of C guarantees that each of its vertices has at least one neighbour in S and at least one neighbour in Y. Since each of u and x has at least one neighbour in S, there is a path from u to x with at least one interior vertex and with all interior vertices in S. Let P be a shortest such path; note that P has no chords except possibly the chord ux. Similarly, there is a path Q from u to x with at least one interior vertex, and with all interior vertices in Y, that has no chords except possibly the chord ux. The union of P and Q is a cycle of length at least four; since this cycle must have a chord, vertices u and x must be adjacent. In turn, the union of Q and ux is a chordless cycle, and so Q has precisely two edges. This means that some vertex v in Y is adjacent to both u and x.

Write $i = \mathit{dist}(s,x)$ and $j = \mathit{dist}(x,y)$. Since all vertices t with $\mathit{dist}(s,t) < i$ belong to S and since v has no neighbours in S, we must have $\mathit{dist}(s,v) > i$; since $\mathit{dist}(x,v) = 1$, we conclude that $\mathit{dist}(s,v) = i+1$ and that $v \in \overline{sx}$. Since $\overline{sx} = \overline{sy}$, it follows that $v \in \overline{sy}$. Since $\mathit{dist}(v,x) = 1$ and $\mathit{dist}(x,y) = j$, we have $\mathit{dist}(v,y) \leq j+1$. From $\mathit{dist}(s,v) = i+1$, $\mathit{dist}(s,y) = i+j$, $\mathit{dist}(v,y) \leq j+1$, $i \geq 1$, $j \geq 1$, and $v \in \overline{sy}$, we deduce that $\mathit{dist}(v,y) = j-1$.

Since $\mathit{dist}(u,v) = 1$, it follows that $\mathit{dist}(u,y) \leq j$; since $\mathit{dist}(s,u) = i$ and $\mathit{dist}(s,y) = i+j$, we conclude that $\mathit{dist}(u,y) = j$ and $u \in \overline{sy}$. Since $\mathit{dist}(s,u) = i$, $\mathit{dist}(s,x) = i$, and $\mathit{dist}(u,x) = 1$, we have $u \notin \overline{sx}$. But then $\overline{sx} \neq \overline{sy}$, a contradiction. □

A vertex of a graph is called *simplicial* if its neighbours are pairwise adjacent.

Lemma 2. *Let s, x, y be three distinct vertices in a finite connected chordal graph. If s is simplicial and $\overline{sx} = \overline{sy}$, then $\overline{xy}$ consists of all the vertices of the graph.*

Proof. Since $\overline{sx} = \overline{sy}$, we have $y \in \overline{sx}$, and so $[ysx]$ or $[syx]$ or $[sxy]$; since s is simplicial, $[ysx]$ is excluded; switching x and y if necessary, we may assume that $[sxy]$. Given an arbitrary vertex u, we have to prove that $u \in \overline{xy}$. Let P be a shortest path from s to u and let Q be a shortest path from u to y. Lemma 1 guarantees that x is a cut vertex separating s and y, and so the concatenation of P and Q must pass through x. This means that $[sxu]$ or $[uxy]$ (or both). If $[uxy]$, then $u \in \overline{xy}$; to complete the proof, we may assume that $[sxu]$, and so $u \in \overline{sx}$.

Since $\overline{sx} = \overline{sy}$, we have $[usy]$ or $[suy]$ or $[syu]$; since s is simplicial, $[usy]$ is excluded. If $[suy]$, then $[sxu]$ implies $[xuy]$; if $[syu]$, then $[sxy]$ implies $[xyu]$; in either case, $u \in \overline{xy}$. □

Proof of Theorem 1. Consider a connected chordal graph on n vertices where $n \geq 2$. By a theorem of Dirac [9], this graph has at least two simplicial vertices; choose one of them and call it s. We may assume that the lines $\overline{sz}$ with $z \neq s$ are pairwise distinct (else some line consists of all n vertices by Lemma 2). Since the graph is connected and has at least two vertices, s has at least one neighbour; choose one and call it u. If u is the only neighbour of s, then every path from s to another vertex must pass through u, and so $\overline{su}$ consists of all n vertices. If s has a neighbour v other than u, then line $\overline{uv}$ is distinct from all of the $n-1$ lines $\overline{sz}$ with $z \neq s$: since s, u, v are pairwise adjacent, we have $s \notin \overline{uv}$. $\square$

3 Related theorems

In Theorem 1, 'connected chordal graph' can be replaced by 'connected bipartite graph':

- every metric space induced by a connected bipartite graph on n vertices, where $n \geq 2$, has a line that consists of all n vertices.

In fact, $\overline{xy}$ consists of all n vertices whenever x and y are adjacent. To prove this, consider an arbitrary vertex u. Since the graph is bipartite, $dist(u, x)$ and $dist(u, y)$ have distinct parities; since $dist(x, y) = 1$, they differ by at most one. We conclude that $dist(u, x)$ and $dist(u, y)$ differ by precisely one, and so $u \in \overline{xy}$.

In Theorem 1, 'connected chordal graph' can be also replaced by 'sufficiently large graph of diameter two': Chiniforooshan and Chvátal [6] proved that

- every metric space on n points where each nonzero distance equals 1 or 2 has $\Omega(n^{4/3})$ distinct lines and this bound is tight.

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